An Analysis of the Draft Utah Math Standards

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Introduction and Conclusions.

In reviewing the new Utah Mathematics Standards I felt it best to focus on sixth and seventh grade. The problems there are representative of the problems in the previous grades and in the separate discussions of Algebra I, Geometry, and Intermediate Algebra. Also, compared to the material in the lower grades, the failings in the new Utah Mathematics Standards document become more evident in sixth and seventh grade. For example, by seventh grade it becomes clear that Utah's requirements are roughly two years behind those of high achieving countries, and the mathematical errors - of which there are a very large number throughout all the lower grades - become more obvious.

A number of the errors that were present in seventh grade and above when I first looked at the document have recently been corrected - apparently in response to the criticism that Prof. H.-H. Wu of Berkeley University gave of those standards. However, in the process the editors have introduced new errors and have not addressed the fundamental problem that these standards are far below international expectations.

My overall conclusion is that though there are a number of correct and well stated standards scattered throughout the Utah Standards, the document cannot simply be edited to achieve anything like a world class level. It must be entirely redone. A much greater focus on the key material is required, and the related standards need to be crafted in a way that makes clear both the content and the concepts related to the content that are required.

I also find it incomprehensible that the excellent review that Utah asked for from Prof. Wu was essentially ignored by the editors of the document in grades K - 6. In fact he informed me that

> "Except for (I think) three or four small instances involving very simple changes in the standards of K-6, such as the change of one word (e.g., 'value' to 'number'), they left intact almost EVERY objection I made. In other words, the mess is still where it was before."

As a result, I am forced to conclude that the main people involved in the editing of the K-6 standards have a very fragile understanding of the mathematics involved in these grades. Consequently, I will be a little more detailed in my discussion of the mathematical issues involved in some of the more egregious errors in the sixth and seventh grade standards than Prof. Wu was in his report.

Note: The standards entitled "Math 7" give a remedial course for students not ready for the regular seventh grade course. The actual seventh grade course is called "Pre-Algebra" in the Utah document. As a result I will not comment on any of the "Math 7" standards below, but will focus on the Sixth Grade Standards and on Pre-Algebra.

General Comments on the Utah Grade 6 and Pre-Algebra Standards.

We begin by looking at the overall structure of the Sixth Grade and Pre-Algebra standards.

First note that there are 5 "Standards," 14 "Objectives," as sub-headings under these standards, and 51 items as sub-headings under the objectives in sixth grade. There are also 5 Standards, 14 Objectives and 49 sub-headings under the objectives in the pre-algebra document. In fact, the sub-headings under the objectives are, effectively, the actual standards and I will focus on them.

Each sub-heading is given in highly compressed form, usually taking, at most, about $\frac{2}{3}$ of a line. As a result, all too often it is literally impossible to figure out what the sub-headings are trying to do. Among those I can understand, a significant number seem to contain serious mathematical errors. Moreover, there often seems to be little connection between the sub-headings and the objectives they are under.

One also should note that when we look at the standards of the high achieving countries there are far fewer standards in these grades. The high achieving countries have pruned out the non-basic material, such as almost all of the standards in data analysis and probability so that students can focus on the material that is essential for supporting their learning of more advanced material later. (Data analysis and probability are important topics. However, to actually say anything substantive about them requires far more mathematics and far more sophistication in mathematics than is available in sixth/seventh grade, and should be reserved for a serious course at the high school level.)

This focus on key topics is now accepted as crucial for improved outcomes by all three major mathematics associations in this country - NCTM, AMS, *American Mathematics Society*, and MAA, *Math Association of America*. The NCTM acknowledges them with its Focal Points, and the AMS, MAA, have, on the MAA web-site a very important document that begins as follows:

> "The value of a mathematical education and the power of mathematics in the modern world arise from the cumulative nature of mathematics knowledge. A small collection of simple facts combined with appropriate theory is used to build layer upon layer upon layer of ever more sophisticated mathematical knowledge. The essence of mathematical learning is the process of understanding each new layer of knowledge and thoroughly mastering that knowledge in order to be able to understand the next layer."

My understanding was that the Utah legislature had asked that the new Utah standards be modeled after the new NCTM Focal Points. These focal points align fairly closely with what is **DONE** in high achieving countries, though the Focal Points go about things in a more leisurely way, so the material in the seventh or eighth grade in the Focal Points will have been done earlier in high achieving countries. Moreover, at least in grades K - 5 little besides the material covered in the NCTM Focal Points (through grade 7) is studied during mathematics instruction in the high achieving countries.

However, in the proposed new Utah standards the material from the Focal Points only makes up a small part of the standards at each of sixth and seventh grade, so the **FOCUS** that is essential to the leading international standards and curricula is impossible to duplicate in courses aligned with the new version of the Utah Standards. (I'm not saying that the existing Utah standards are any better than the proposed new standards - they are, if anything, even worse - but what I am saying is that if Utah wants to see significant improvements in student outcomes in mathematics and mathematics related topics, then these new standards have to be completely redone.)

Detailed Comments on the Utah Grade 6 and Pre-Algebra Standards.

The authors of the new Utah standards had the advice of only a very small number of mathematicians. From what I understand, the majority of the committee routinely ignored much of that advice, and this shows. What follows is just one example. The second of the three seventh grade focal points is

> "Measurement and Geometry and Algebra: Developing an understanding of and using formulas to determine surface areas and volumes of three-dimensional shapes."

In the discussion included with this focal point in the NCTM document is the very carefully written sentence:

> "Students see that the formula for the area of a circle is plausible by decomposing a circle into a number of wedges and rearranging them into a shape that approximates a parallelogram."

However, in the sixth grade standards we find Standard IV, Objective 1(d)

> "Decompose a circle into a number of wedges and rearrange the wedges into a shape that approximates a parallelogram to develop the formula for the area of a circle."

In the Focal Points the key word in the sentence is *plausible*. Note that it is entirely absent in the sixth grade standards. Instead students are to *develop the formula for the area of a circle*. Well, flatly, they can't! In order to do this one needs the concept of a limit, and hence the beginnings of the calculus. But people who know very little about mathematics typically make these kinds of mistakes. Such errors are everywhere in the new Utah document.

Here are another series of really bad errors. The sixth grade Standard I, Objective

1(a) reads:

> "Recognize a rational number as a ratio of two integers, a to b where b is not equal to zero."

Prof. Wu strenuously objected to this standard in his report, but his objection was ignored. So let me spell out in more detail what is going on mathematically. Unfortunately, a ratio is not really a number. A ratio really is a line through the origin in the coordinate plane. (We say that two distinct pairs of real numbers, (a, b) and (c, d) with $a^2 + b^2$ and $c^2 + d^2$ non-zero, represent the same ratio if and only if they both lie on the same line through the origin.) We only obtain an identification of all ratios but one with fractions through the use of slope to distinguish the lines other than the y-axis through the origin. So the authors of the Utah standards are mixing apples and oranges, and in the process, attempting to guarantee that the typical total confusion among K - 12 educators about ratios, rates and proportions will be propagated to the next generations of students.

In the original Pre-Algebra standards this error was not present, but it was added to the new, corrected, version. Now the Pre-Algebra document contains Standard II Objective 1(c):

> "Represent percents as ratios based on 100 and decimals as ratios based on powers of 10."

Decimals are simply fractions with denominator a power of 10. But we are being told here that decimals are ratios - lines through the origin. Moreover, "ratios based on 100" is meaningless. I think the authors meant to say that a percent is a fraction of the form $\frac{a}{100}$, something that is also incorrect, but that one finds in any number of textbooks in this country. In fact, a can be any real number, and a percent simply means the ratio represented by the line through the origin with slope $\frac{a}{100}$. It's also worth noting that Standard II 1(a):

> "Compare ratios to determine if they are equivalent,"

again is mixing apples and oranges. A ratio is already an equivalence class - of the points on the line through the origin that represents the ratio. The authors are again confusing ratios with fractions and "equivalent fractions" with "equivalent ratios," but this latter concept simply makes no sense. Likewise, I can't make any sense of II 1(b):

> "Compare ratios using the unit rate."

Since ratios are really lines through the origin, about all we can determine is if we are dealing with different lines or the same line. There is no way of saying one ratio is *bigger* than another any more than we can say one line is bigger than another.

I am not saying that we should introduce ratios as lines through the origin in the early grades, but we should **never** tell students incorrect things about mathematics. What can and should be done in the lower grades is to give accurate descriptions of the properties of ratios, and assure students that later, when they have more tools available, ratios will be firmly grounded for them. That was the way logarithms were handled when I was in fifth and sixth grade and we learned to use logarithm tables for multiplying and raising numbers to powers.

It is worth mentioning that there is a proposed Utah sixth grade standard, I.6(b),

> "Add, subtract, multiply, and divide fractions and mixed numbers."

This is a weak standard because it does not specify that these operations should be entirely fluent. In the first of the third grade Focal Points:

> "Number and Operations and Algebra: Developing understandings of multiplication and division and strategies for basic multiplication facts and related division facts,"

we have the requirement that

> "Students understand the meaning of multiplication and division."

Fluency with division and multiplication of whole numbers is assumed by the end of grade 5, and fluency of addition and subtraction of fractions and decimals is also assumed by the end of grade 5. By the end of fourth grade the Focal Points are asking for exactly the level of competence with the four operations on fractions that is in sixth grade in the new Utah Standards!

In seventh grade we have I.1(a):

> "Compute fluently using all four operations with integers, and explain why the corresponding algorithms work."

This is getting there. It refers to integers, not rationals, so the new ingredient is the four operations on negative integers.

Of course, to this point there has been little discussion in these standards of negative numbers. In particular, though one finds mention of the "additive inverse" in grade 6, there is no discussion of how one multiplies two negative integers, let alone the highly mysterious rule $(-1) \times (-1) = 1$.

In states where careful thought has been given to the sequence of critical steps students have to take to move from whole numbers and positive fractions to integers and rational numbers, there is usually careful attention paid to the problems inherent in teaching the multiplication and division of negative numbers in a mathematically coherent way.

The next seventh grade standard I.1(b) reads:

"Compute fluently using all four operations with rational numbers, including negative fractions and decimals, and explain why the corresponding algorithms work."

The problem here is that there are no "corresponding algorithms." There are definitions of addition and subtraction for fractions -

$$\frac{a}{b} \pm \frac{c}{d} \;=\; \frac{ad \pm bc}{bd},$$

and a definition of fraction multiplication,

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bc}.$$

(Both definitions are conspicuously absent in the Utah standards incidentally.) Once one has these **definitions**, then one simply applies them and uses the ordinary algorithms for integer arithmetic to make everything explicit.

The sixth grade Standard I, Objective 2(b) reads

"Compare and order rational numbers, including positive and negative mixed fractions and decimals, using a variety of methods, including the number line and finding common denominators."

I find it astounding that the KEY method of comparing fractions, cross multiplying and comparing the cross products, is not mentioned here. Putting the fractions over a common denominator actually involves considerably more work than just cross multiplying.

Another problematic sixth grade standard is I.4(b):

"Recognize that ratios derive from pairs of rows in the multiplication table and connect with equivalent fractions."

This is taken by the authors of new Utah Standards from the discussion in the Focal Points document. But the main authors of the Utah document do not appear to have sufficiently understood what the Focal Points was actually saying here. What was meant there was that if you select any two rows in the multiplication table and compare the entries in the corresponding columns, then the pairs in any two columns are in the same ratio, so in the $\times 2$ and $\times 5$ rows,

$$\begin{pmatrix} 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\ 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 \end{pmatrix}$$

we have that $\frac{2}{5} = \frac{4}{10} = \cdots = \frac{18}{45}$.

in other words, each pair lies on the same line through the origin. Ratios do not "derive" from the rows of the multiplication table, but the rows of the multiplication table give excellent examples of numbers that are in the same ratio, and, as a result, the quotients taken in the same order for each pair, represent the same fraction.

There are many other areas where these standards are very problematic, far too many for me to be exhaustive even when focusing on just two grades. I will mention just one more. The sixth grade Standard II.1(b) reads:

> "Draw a graph and write an equation from a table of values."

The issue here is that a table of values is, of necessity, finite. So the only graph one can draw consists of a finite number of points on the coordinate plane. As a matter of logic, it is difficult to see how such a finite number of points can give rise to an equation since, equations, as they are understood in constructing tables, refer to things like $3x + 2y = 4$ that are true for an infinite number of values of the pairs of variables (x, y).

Conclusions.

As I said, I've just scratched the surface here. Prof. Wu's description of the document as "the mess" is entirely apt.

It has been my experience that when standards do not spell out, in detail, what needs to be covered, that material will not be covered. Additionally, when there is no coherence to the standards, there will be no coherence in instruction. Students will simply learn long lists of factoids, and will never develop anything approaching mathematical proficiency.

It has also been my experience that when the understanding of school mathematics in a state's standards is seriously flawed, as is the case in these standards, then students continue to learn incorrect things and find that it is extremely difficult, if not impossible, to ever achieve the level of competence in the subject and in any of those areas that depend on mathematics that is required of todays workers and leaders.

So I am forced to conclude, as I stated in the introduction, that it is impossible to simply revise the Utah document. It must be entirely redone.